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Solitary wave solutions of the generalised Burgers–Huxley equation

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Abstract. Exact solitary wave solutions of the generalised Burgers-Huxley equation

$$\frac{\partial u}{\partial t} - \alpha u^\delta \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u^\delta)(u^\delta - \gamma)$$

are obtained by using the relevant nonlinear transformations. The results obtained are the generalisation of former work. The method in this paper can also be applied to the Burgers-Fisher equation.

In this paper we consider exact solitary wave solutions of the following nonlinear diffusion equation:

$$\frac{\partial u}{\partial t} + \alpha u^\delta \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u^\delta)(u^\delta - \gamma) \tag{1}$$

where α, β, γ and δ are parameters, $\beta \geqslant 0$, $\delta > 0$, $\gamma \in (0, 1)$. Equation (1) is an extended form of the famous Huxley and Burgers equations. When $\alpha = 0$, $\delta = 1$, equation (1) is reduced to the Huxley equation which describes nerve pulse propagation in nerve fibres and wall motion in liquid crystals [1, 2]

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u)(u-\gamma). \tag{2}$$

When $\beta = 0$, $\delta = 1$, equation (1) is reduced to the Burgers equation which describes the far field of wave propagation in nonlinear dissipative systems [3]

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{3}$$

It is known that nonlinear diffusion equations (2) and (3) play important roles in nonlinear physics. They are of special significance for studying nonlinear phenomena. If we take $\delta = 1$ and $\alpha \neq 0$, $\beta \neq 0$, (1) becomes the following Burgers–Huxley equation:

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u)(u-\gamma). \tag{4}$$

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Equation (4) shows a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transport. This equation was investigated by Satsuma [4] in 1986. By using the Hirota method in soliton theory, solitary wave solutions of equation (4) were obtained. Exact solutions of nonlinear differential equations are of importance in physical problems. So far there exists no general method for finding solutions of nonlinear diffusion equations. Generally, a relevant nonlinear transformation is a powerful method for solving nonlinear differential equations. It is well known that the Burgers equation may be exactly solved by using the Cole–Hopf transformation, which is an enlightening example. In this paper we present the solitary wave solutions of equation (1) via the introduction of nonlinear transformations. Such transformations are simple but very effective.

In order to obtain the solution of equation (1), we first make the transformation

$$u = v^{1/\delta}. \tag{5}$$

Equation (1) becomes

$$\frac{\partial v}{\partial t} + \alpha v \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} - \left(\frac{1}{\delta} - 1\right)\frac{1}{v}\left(\frac{\partial v}{\partial x}\right)^2 = \delta\beta v(1-v)(v-\gamma). \tag{6}$$

Then let $v(x, t) = v(x - ct) = v(\xi)$; so (6) becomes

$$-c\frac{\mathrm{d}v}{\mathrm{d}\xi} + \alpha v \frac{\mathrm{d}v}{\mathrm{d}\xi} - \frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - \left(\frac{1}{\delta} - 1\right)\frac{1}{v}\left(\frac{\mathrm{d}v}{\mathrm{d}\xi}\right)^2 = \delta\beta v(1-v)(v-\gamma). \tag{7}$$

We make the following ansatz:

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = av(v-\gamma) \tag{8}$$

where a is an undetermined parameter. So we should have

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} = a^2(2v-\gamma)v(v-\gamma). \tag{9}$$

According to (8) and (9), equation (7) becomes

$$\begin{aligned} &-cav(v-\gamma) + \alpha vav(v-\gamma) - a^2(2v-\gamma)v(v-\gamma) \\ &\qquad -\left(\frac{1}{\delta} - 1\right)\frac{1}{v}[av(v-\gamma)]^2 = \delta\beta v(1-v)(v-\gamma). \end{aligned} \tag{10}$$

A comparison between the corresponding terms in equation (10) gives the value of the parameters a and c

$$a = \frac{1}{2}\left\{\frac{\alpha\delta}{1+\delta} \pm \left[\left(\frac{\alpha\delta}{1+\delta}\right)^2 + \frac{4\delta^2\beta}{1+\delta}\right]^{1/2}\right\} \tag{11}$$

$$c = \frac{\gamma}{\delta}a - \frac{\delta\beta}{a} = \frac{\gamma\alpha}{1+\delta} - \frac{1+\delta-\gamma}{2(1+\delta)}\left(-\alpha \pm \sqrt{\alpha^2 + 4\beta(1+\delta)}\right). \tag{12}$$

Integration of equation (8) gives

$$v = \frac{\gamma}{1 + \exp(\gamma a\xi + b)} \tag{13}$$

where b is an integration constant. Coming back to equations (5), (11), (12), we get the solitary wave solutions of the generalised Burgers-Huxley equation (1)

$$u(x,t)=\left[\!\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left\{\frac{-\alpha\delta\pm\delta\sqrt{\alpha^2+4\beta(1+\delta)}}{4(1+\delta)}\gamma\right.\right.\right.$$
$$\left.\left.\left.\times\left[x-\left(\frac{\gamma\alpha}{1+\delta}-\frac{(1+\delta-\gamma)(-\alpha\pm\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}\right)t\right]\right\}\right]\!\right]^{1/\delta}. \tag{14}$$

If we let $\beta=0$, the above solution becomes

$$u(x,t)=\left\{\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left[\frac{-\alpha\delta\gamma}{2(1+\delta)}\left(x-\frac{\gamma\alpha}{1+\delta}t\right)\right]\right\}^{1/\delta}. \tag{15}$$

This is the solution of the generalised Burgers equation

$$\frac{\partial u}{\partial t}+\alpha u^\delta\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=0. \tag{16}$$

If we let $\alpha=0$, solution (14) becomes

$$u(x,t)=\left\{\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left[\pm\frac{\delta\gamma}{2}\sqrt{\frac{\beta}{1+\delta}}\left(x-\frac{1+\delta-\gamma}{1+\delta}\sqrt{\beta(1+\delta)}\,t\right)\right]\right\}^{1/\delta}. \tag{17}$$

This is the solution of the generalised Huxley equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=\beta u(1-u^\delta)(u^\delta-\gamma). \tag{18}$$

Note that we can also make the ansatz

$$\frac{\mathrm{d}v}{\mathrm{d}\xi}=av(1-v). \tag{19}$$

By using the same steps, another solitary wave solution connecting the two stationary states, $u=0$ and $u=1$, is easily obtained:

$$u(x,t)=\left[\!\left[\tfrac{1}{2}+\tfrac{1}{2}\tanh\left[\frac{-\alpha\delta\pm\delta\sqrt{\alpha^2+4\beta(1+\delta)}}{4(1+\delta)}\right.\right.\right.$$
$$\left.\left.\left.\times\left[x-\left(\frac{\alpha}{1+\delta}-\frac{(1-\gamma-\delta\gamma)(\alpha\pm\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}\right)t\right]\right\}\right]\!\right]^{1/\delta}. \tag{20}$$

The exact solutions obtained here describe the special coherent structures in the corresponding nonlinear dissipative systems. Clearly the method presented here to obtain solitary wave solutions of nonlinear diffusion equations is indeed simple and effective. Finally we would like to point out that the same approach gives the solitary wave solutions of the following nonlinear equation:

$$\frac{\partial u}{\partial t}+\alpha u^\delta\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=\beta u(1-u^\delta). \tag{21}$$

Equation (22) may be called the generalised Burgers-Fisher equation. The solutions we obtained are

$$u(x,t)=\left[\!\left[\tfrac{1}{2}+\tfrac{1}{2}\tanh\left\{\frac{-\alpha\delta}{2(\delta+1)}\left[x-\left(\frac{\alpha}{\delta+1}+\frac{\beta(\delta+1)}{\alpha}\right)t\right]\right\}\right]\!\right]^{1/\delta}. \tag{22}$$

In the special case $\delta=1$, solution (22) is reduced to the result obtained in [4].

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